Engineering Notes

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Expansion of the Third-Body Disturbing Function

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Introduction

THE expansion of the disturbing function still remains one of the most important problems in celestial mechanics, both in planetary applications and artificial satellite problems. The expansion of the gravitational disturbing function in zonal and tesseral harmonics based on the Legendre polynomials is now relatively easy to perform with the help of modern digital computers.

On the other hand, the disturbing function for the third-body effects (solar and lunar perturbations on artificial satellites) is more difficult to expand, even with the availability of fast digital computers. The classically known expansions come in two categories: with the Laplace coefficients[1] and with the Legendre polynomials.[2-7]

In the present Note we develop a new form of the third-body disturbing function which is in the category of the Legendre expansions. The purpose of this work is to find an expansion which is especially efficient and easy to carry out to a high order with the use of a package of Poisson series programs for algebraic operations on a computer.

The reader should be aware of the fact that the most efficient expansion for hand calculations is usually not the most efficient expansion with a computerized algebraic processor. Simplicity of programming is another factor that needs to be considered on a computer. In general, the computer will prefer simple recurrence relations. The Poisson series processor will usually have a certain number of standard operations available: repeated partial differentiation, substitution of a series for a polynomial variable, the bionomial theorem, and the Bessel series of the Kepler problem are all standard one-line operation with our processor. It is clear that many of the classical celestial mechanics problems must be completely reformulated in the light of these modern tools.

The expansion of the disturbing function, which is the object of the present text, is a literal expansion of the following general form:

$$R=\sum_{n}^{\infty}\epsilon^{n}\left(X_nX_n^{(3)}+Y_nY_n^{(3)}\right)$$

with the remarkable property that it is separated in factors each depending only on one body. In other words, X_n depends only on the elements (e, i, M, ω, Ω) while $X_n^{(3)}$ depends only on the elements ($e_3,i_3,M_3,\omega_3,\Omega_3$) of the third (perturbing) body. These factors are therefore Poisson series in five variables only. They are relatively short series, in comparison with the resulting series R which is an expansion in ten variables. Here ϵ represents the ratio of semimajor axes (a/a').

The following section develops the details of the expansion. The main result is in fact an expansion of the third-body effects in time-dependent tesseral harmonics.

Expansion with Legendre Polynomials

The classical expression that is known for the disturbing function is as follows:

$$R=\mu_3\left[\frac{1}{\Delta}-\frac{\vec{r}\cdot\vec{r}_3}{r_3^3}\right]=\frac{\mu_3}{r_3}\sum_{n=2}^{\infty}\left(\frac{r}{r_3}\right)^nP_n(\cos\psi)$$

Here, μ_3 is the constant $Gm_3=k^2m_3$ related to the mass m_3 of the third, or disturbing, body. Also, Δ is the distance between the satellite and the disturbing body:

$$\Delta^2=r^2+r_3^2-2rr_3\cos\psi=|\vec{r}_3-\vec{r}|^2$$

The vectors $\vec{r}_3$ and $\vec{r}$ are the position vectors of the third body and the satellite relative to the central body (the Earth in most artificial satellite applications), ψ being the angle between these two vectors, so that we may write

$$\vec{r}\cdot\vec{r}_3=rr_3\cos\psi$$

In order to obtain a better expression for $\cos\psi$, we first introduce the coordinates of the different bodies in terms of their radius vector r, right ascension α, and declination δ:

$$\vec{r}=\begin{cases}x=r\cos\delta\cos\alpha\\ y=r\cos\delta\sin\alpha\\ z=r\sin\delta\end{cases}\qquad \vec{r}_3=\begin{cases}x_3=r_3\cos\delta_3\cos\alpha_3\\ y_3=r_3\cos\delta_3\sin\alpha_3\\ z_3=r_3\sin\delta_3\end{cases}$$

so that

$$\cos\psi=\sin\delta\sin\delta_3+\cos\delta\cos\delta_3\cos(\alpha-\alpha_3)$$

We can now apply the standard addition theorem of Legendre polynomials, so that[8-10]

$$P_n(\cos\psi)=\sum_{m=0}^{n}W_{nm}P_{nm}(\sin\delta)P_{nm}(\sin\delta_3)\cos m(\alpha-\alpha_3)$$

with

$$W_{nm}=(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}$$

where δ_{0m} is a Kronecker delta and P_{nm} are *associated* Legendre functions. It will be seen that the usefulness of this formula comes essentially from the fact that each one of the two factors P_{nm} depends on only *one* of the two bodies.

In order to simplify this expression still further, we introduce the modified associated Legendre functions Q_{nm}, defined by

$$Q_{nm}(x)=\frac{d^m}{dx^m}P_n(x)$$

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Then

$$P_{nm}(x) = (1-x^2)^{m/2} Q_{nm}(x)$$

or

$$P_{nm}(\sin\delta) = \cos^m\delta \cdot Q_{nm}(\sin\delta)$$

with a similar expression in δ_3.

Thus, the principal factor

$$P_{nm}(\sin\delta) P_{nm}(\sin\delta_3)\cos m(\alpha-\alpha_3)$$

can be written as

$$Q_{nm}(\sin\delta) Q_{nm}(\sin\delta_3)\cos^m\delta\cos^m\delta_3$$
$$\times(\cos m\alpha\cos m\alpha_3 + \sin m\alpha\sin m\alpha_3)$$

At this point, we introduce new functions C_m and S_m, which will allow us to write the third-body potential R in a completely separated form. We have

$$r^2 = x^2+y^2+z^2 \qquad \tilde{r}^2 = x^2+y^2 = r^2\cos^2\delta$$

so that

$$\sin\delta = z/r \qquad \cos\delta = \tilde{r}/r$$

This suggsts that we define the new dimensionless functions as follows:

$$C_m(\alpha,\delta) = \cos^m\delta\cos m\alpha = (\tilde{r}/r)^m\cos m\alpha$$

$$S_m(\alpha,\delta) = \cos^m\delta\sin m\alpha = (\tilde{r}/r)^m\sin m\alpha$$

which may be generated recursively starting with the initial values $C_0=1$, $S_0=0$, $C_1=x/r$, $S_1=y/r$, and the obvious recurrence relations:

$$C_m = C_1C_{m-1} - S_1S_{m-1} \qquad S_m = C_1S_{m-1} + S_1C_{m-1}$$

With the use of the C_m and the S_m which are functions of α and δ, as well as the corresponding $C_m^{(3)}$ and $S_m^{(3)}$ which are functions of α_3 and δ_3 for the perturbing body, we have

$$R = \frac{\mu_3}{r_3}\sum_{n=2}^{\infty}\left(\frac{r}{r_3}\right)^n\sum_{m=0}^{n} W_{nm}Q_{nm}(\sin\delta)Q_{nm}(\sin\delta_3)$$
$$\times[C_mC_m^{(3)} + S_mS_m^{(3)}]$$

Finally, we make one more simplification for the purpose of grouping the factors involving the satellite orbit elements (excluding all other factors which only involve the perturbing body). This is in fact the most important step of all the operations. The disturbing function can be efficiently expanded to very high orders in the elements if the factors involving each body are explicitly separated. In order to achieve this separation, we replace the factor (μ_3/r_3) $(r/r_3)^n$ which is present in R, by a product of four factors:

$$\frac{\mu_3}{a_3}\left[\frac{a}{a_3}\right]^n\left(\frac{r}{a}\right)^n\left(\frac{a_3}{r_3}\right)^{n+1}$$

the last three of which are dimensionless. The constant factor $\epsilon = a/a_3$ is the small parameter of the expansion. The last two factors have well-known series representations in terms of Bessel functions. As was pointed out previously, each of these series involves only *one* of the two bodies. We also note the two *different* exponents, n and $n+1$.

The disturbing function can now be written as

$$R = \frac{\mu_3}{a_3}\sum_{n=2}^{\infty}\sum_{m=0}^{n} W_{nm}\epsilon^n[X_{nm}X_{nm}^{(3)} + Y_{nm}Y_{nm}^{(3)}]$$

The following elementary two-body functions, called surface harmonics, have also been defined:

$$X_{nm} = (r/a)^n Q_{nm}(\sin\delta) C_m(\alpha,\delta)$$

$$X_{nm}^{(3)} = (a_3/r_3)^{n+1} Q_{nm}(\sin\delta_3) C_m(\alpha_3,\delta_3)$$

The Y-factors are similar, except that they use the S_m-functions. This is what we call the separated form of the disturbing function: all the terms have been decomposed in two factors, each factor depending on only *one* of the two bodies, either the satellite *or* the perturbing body. These factors can be expanded separately in Poisson series. These series are relatively short, considering that they contain only a small number of orbit elements and angles. Therefore, we are here in the presence of one of the most efficient forms of expansion of the third-body disturbing function.

It is seen that the C-series and S-series are in cosines and sines, respectively, with the same coefficients. It would therefore be advantageous to define a complex series $\tilde{C}_m = C_m + iS_m$ incorporating both the cosine and sine terms.

We also see that a complex $X_{nm} + iY_{nm} = \tilde{X}_{nm}$ should be defined, for the same reason as mentioned above. If this would be done, we would then have the formula

$$P_n(\cos\psi) = R_e(\tilde{C}_m\bar{\tilde{X}}_{nm})$$

where the horizontal bar indicates the complex conjugate.

The most important conclusion that can be drawn from this study is the following: the series C_m, S_m, X_{nm}, and Y_{nm} containing only the *two* angles α and δ are all very short—much shorter than the series $P_n(\cos)$ containing all four angles. For this reason it is recommended that the series depending on (α,δ) *never* be multiplied explicitly by the series containing α_3, δ_3. It is possible, and in fact more efficient, to keep the factors "unmultiplied" in about all the important applications such as taking partial derivatives, deriving equations of motion, and performing averaging.

The most useful property of this form of the potential is probably related to the *averaging* operations in order to obtain secular effects. To average over the satellite period, we only have to average X_{nm} and Y_{nm}. To average over the third-body period, we only have to average $X^{(3)}$ and $Y^{(3)}$. The double average is obtained by successively performing the two single averages.

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Airborne Method to Minimize Fuel with Fixed Time-of-Arrival Constraints

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Introduction

THE world's transportation system is being adversely affected by the decreasing supply of petroleum coupled with the increasing demand for travel and shipment of cargo. This situation especially affects the air transportation industry where there is no current alternative to jet fuel. Thus, it has become mandatory to seek new ways to conserve fuel by building more energy-efficient aircraft and by flying existing aircraft along more fuel-efficient flight paths.

One concept of near-term impact is application of trajectory optimization principles to generate the vertical profile (and the inherent elevator/throttle control sequence) to minimize fuel and other costs in flying over a fixed horizontal path (between a fixed city pair). This Note focuses on the process for meeting destination time-of-arrival constraints and the potential benefits that an on-board flight management system having this capability might have.

It is assumed that 1) aircraft will soon be nominally flown along minimum-fuel vertical flight paths, 2) increasing delays of variable length will occur at the arrival point because of expected congestion, and 3) the air traffic control (ATC) system can anticipate this congestion and inform the pilot what his delay will be before he arrives in the terminal area (i.e., the controller assigns the pilot an expected time-of-arrival). The pilot has a choice of two strategies to follow: 1) continue to fly his nominal minimum fuel path and then go into a minimum-fuel-rate holding pattern to absorb the delay at the end of the cruise segment; or 2) slow down so that he arrives at the terminal area within an acceptable tolerance of the assigned time-of-arrival.

The algorithm developed in this study generates the vertical flight path between a city pair which minimizes fuel and meets the delayed time-of-arrival constraint of option 2 above. The fuel reduction of using this strategy is compared to that of option 1 as a function of delay time. Further detail can be found in Ref. 1.

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Previous significant work in generating optimum vertical flight paths that minimize fuel or direct operating costs are summarized in Ref. 2. Schultz and Zagalsky developed an algorithm that solved the minimum-fuel, fixed-range, fixed-time problem.[3] Erzberger and his colleagues extended this algorithm to minimize direct operating costs in the presence of winds.[4] They also developed efficient computer programs for generating typical flight profiles.[5] This study represents an extension of Erzberger's work to include control of time-of-arrival.

Approach

The longitudinal point mass model of the aircraft can be described with five state variables: airspeed V, flight-path angle γ, altitude h, horizontal range x, and mass m (Ref. 4). For optimization purposes, it is adequate to combine h and V into the specific energy state

$$E=h+V^2/2g \tag{1}$$

with its time derivative

$$\dot{E}=V(T-D)/mg \tag{2}$$

The rate of change of γ is rapid, relative to the other state variables, so that it can be considered as a control. Also, the mass burn rate can be ignored in optimizing the flight path. This leaves two state equations for range and energy.

The cost function to be minimized is of the form

$$J=\int_{t_0}^{t_f}(K_f\dot{f}+K_t)\,\mathrm{d}t \tag{3}$$

where K_f and K_t are the unit costs of fuel (\$/lb) and time (\$/h), and $\dot{f}$ is the fuel flow rate. The problem then is to choose the sequence of controls that satisfy the constraints and minimize the cost of flight governed by Eq. (3).

Erzberger uses the assumption that aircraft energy monotonically increases during climb and monotonically decreases during descent.[4] Energy E is then used as the independent variable by dividing Eq. (2) into Eq. (3). This result produces a constant value for the adjoint variable λ on an optimum trajectory. This variable is the cost of flight during cruise expressed as

$$\lambda=(K_f\dot{f}+K_t)/(V+V_w) \tag{4}$$

where V_w is the longitudinal component of windspeed. The Hamiltonian that is minimized during climb and descent is

$$\min_{\pi,V}\frac{K_f\dot{f}+K_t-\lambda(V+V_w)}{(T-D)V/mg} \tag{5}$$

where controls are throttle setting π and airspeed. Equation (5) is incorporated into the flight management system as the algorithm for on-board generation of the near-optimum flight path.

If the cost of time K_t in Eqs. (4) and (5) is set to zero, the result is the minimum fuel flight path. If this produces the nominal time-of-arrival, then to realize arrival delays beyond this point requires that the coefficient K_t be set negative. The minimum value of λ occurs where cruise airspeed produces minimum fuel rate $\dot{f}_{\min}$. This is found by setting the derivative of Eq. (4) with respect to V to zero and by using the fact that $\dot{f}$ is minimum where $\partial\dot{f}/\mathrm{d}V$ is zero. This condition yields

$$K_{t_{\min}}=-K_f\dot{f}_{\min} \tag{6}$$

Using these values of (K_f,K_t) and λ of zero in Eq. (5) produces the maximum practical delay time that can be achieved by reduced flight speed. Further flight delay should